

# Spherical roots of spherical varieties

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ABSTRACT. Brion proved that the valuation cone of a complex spherical variety is a fundamental domain for a finite reflection group, called the little Weyl group. The principal goal of this paper is to generalize this theorem to fields of characteristic unequal to 2. We also prove a weaker version which holds in characteristic 2, as well. Our main tool is a generalization of Akhiezer's classification of spherical varieties of rank 1.

## 1. Introduction

Let  $G$  be a connected reductive group defined over an algebraically closed field  $k$  of arbitrary characteristic  $p$ . A  $G$ -variety  $X$  is *spherical* if the Borel subgroup  $B$  of  $G$  has an open orbit in  $X$ . For  $p = 0$  there exists a well-developed structure theory for spherical varieties. The present paper is part of a program to generalize this structure theory to arbitrary characteristic.

A crucial part of characteristic zero theory depends on Akhiezer's list, [Akh83], of spherical varieties of rank 1. In the companion paper [Kno13] we compiled results which can be proved without such a list. In the present paper we use Akhiezer's list, after generalizing it to arbitrary characteristic, to prove Brion's theorem on the structure of the valuation cone.

More precisely, for  $p = 0$ , Brion proved in [Bri90] that the valuation cone of a spherical variety is the fundamental domain of a finite reflection group, the little Weyl group  $W_X$  of  $X$ . Following Brion, we define  $W_X$  to be the group generated by the reflections about the codimension-1-faces of the valuation cone  $\mathcal{V}(X)$ . Our first important result, [Theorem 4.3](#), states that  $W_X$  is always finite. This entails immediately that  $\mathcal{V}(X)$  is always a union of Weyl chambers of  $W_X$ .

Next, we investigate when  $\mathcal{V}(X)$  consists of just one chamber. Since counterexamples were known by Schalke [Sch11] for  $p = 2$  it came a bit as a surprise that  $\mathcal{V}(X)$  is indeed a single Weyl chamber whenever  $p \neq 2$  ([Corollary 4.7](#)).

There is even a version of Brion's theorem which is valid for arbitrary, possibly non-spherical,  $G$ -varieties. In that case, one considers the set  $\mathcal{V}_0(X)$  of  $G$ -invariant valuations of  $k(X)$  which are trivial on the subfield  $k(X)^B$ . Then again,  $\mathcal{V}_0(X)$  is the fundamental domain for a unique reflection group  $W_X$ , provided that  $p \neq 2$ . As a matter of fact, this statement is a formal consequence of the spherical case (this was already observed in [Kno93]).

Back to the spherical case: in characteristic zero there are, besides Brion's, several approaches to the little Weyl group. Unfortunately, they all use Lie algebra techniques which do not carry over to positive characteristic. Also, it seems to be hard to make

Brion's original proof work for  $p \neq 0$ . But still, we follow his proof in “spirit” in that we carefully investigate the dihedral angles of  $\mathcal{V}(X)$  and that we use case-by-case arguments.

More precisely, we study the normal vectors to the codimension-1-faces of  $\mathcal{V}(X)$ . Properly normalized they are called the *spherical roots* of  $X$ . This is now where the rank-1-varieties come in: they provide us with all possible spherical roots. For  $p = 0$  their classification was achieved by Akhiezer, [Akh83]. For arbitrary  $p$  we follow mostly a simplification due to Brion, [Bri89]. It turns out (see the table §7) that there are no surprises: all cases are known from characteristic zero or can be reduced to a known one using an inseparable isogeny.

Using this table it is not difficult to see that the angle of any two spherical roots of  $X$  is almost always obtuse. Then, for ruling out the few exceptions, we use the structure theory developed in [Kno13]. For  $p = 2$ , some cases remain which are all listed in Theorem 4.5.

**Acknowledgment:** I would like to thank Guido Pezzini for many discussions on the matter of this paper.

**Notation:** In the entire paper, the ground field  $k$  is algebraically closed. Its characteristic exponent is denoted by  $p$ , i.e.,  $p = 1$  if  $\text{char } k = 0$  and  $p = \text{char } k$ , otherwise. The group  $G$  is connected reductive,  $B \subseteq G$  is a Borel subgroup, and  $T \subseteq B$  is a maximal torus. Let  $\Xi(T) = \Xi(B)$  be its character group. The set of simple roots with respect to  $B$  is denoted by  $S \subset \Xi(T)$ . For  $\alpha \in S$  let  $P_\alpha \subseteq G$  be the corresponding minimal parabolic subgroup. The Weyl group of  $G$  with respect to  $T$  is  $W$ .

For a spherical variety  $X$  let  $\Xi(X) \subseteq \Xi(T)$  be the group of weights of  $B$ -semiinvariant rational functions on  $X$ . By definition, the rank of  $X$  is  $\text{rk } X := \text{rk } \Xi(X)$ . We also use the variants  $\Xi_{\mathbb{Q}}(X) = \Xi(X) \otimes \mathbb{Q}$  and  $\Xi_p(X) = \Xi(X) \otimes \mathbb{Z}_p$  with  $\mathbb{Z}_p := \mathbb{Z}[\frac{1}{p}]$ . For  $\alpha \in S$  let  $\alpha^r$  be the linear function  $\chi \mapsto \langle \chi, \alpha^\vee \rangle$  restricted to  $\Xi_{\mathbb{Q}}(X)$ .

A subgroup  $H \subseteq G$  is called spherical if  $X = G/H$  is spherical. In that case, we call  $\text{rk } X$  the *corank* of  $H$ .

The set  $\mathcal{V}(X)$  of  $G$ -invariant valuations of  $X$  can be considered as a subset of the dual space  $N_{\mathbb{Q}}(X) = \text{Hom}(\Xi_{\mathbb{Q}}(X), \mathbb{Q})$  (see [Kno91, Cor. 1.8]). It is known to be a finitely generated convex cone ([Kno91, Cor. 5.3]). A *spherical root* of  $X$  is a primitive element  $\sigma \in \Xi_p(X) \cap \mathbb{Z}S$  such that  $\sigma$  is non-positive on  $\mathcal{V}(X)$  and  $\mathcal{V}(X) \cap \{\sigma = 0\}$  is one of its codimension-1-faces. The set of spherical roots is denoted by  $\Sigma(X)$ .

Let  $Bx_0 \subseteq X$  be the open  $B$ -orbit. The irreducible components of  $Gx_0 \setminus Bx_0$  are called the *colors* of  $X$ . We say that  $\alpha \in S$  is of type  $(p)$  if  $P_\alpha x_0 = Bx_0$ . The set of simple roots  $\alpha$  which are of type  $(p)$  is denoted by  $S^{(p)}(X)$  or simply  $S^{(p)}$ .

## 2. Classification of spherical varieties of rank one

The aim of this section is to state the classification of (reduced) spherical subgroups  $H \subseteq G$  with  $\text{rk } G/H = 1$ . Thereby, we get also a list of all possible spherical roots. In characteristic zero, this has been first achieved by Akhiezer [Akh83]. Here, we follow closely a simplification due to Brion [Bri89].

We start by describing the process of (parabolic) induction. For that let  $P \subseteq G$  be a parabolic subgroup,  $G_0$  a connected reductive group and  $\pi : P \twoheadrightarrow G_0$  a surjective

homomorphism. Let  $X_0$  be a  $G_0$ -variety. Via  $\pi$ , one may consider  $X_0$  as a  $P$ -variety. Then  $X = G \times^P X_0$  is called the  *$G$ -variety induced from the  $G_0$ -variety  $X_0$  (via  $\pi$ )*. In practice, it is convenient to induce from a parabolic subgroup  ${}^-P$  which is opposite to the chosen Borel subgroup  $B$ . The homomorphism from  ${}^-P$  to  $G_0$  is still denoted by  $\pi$ .

The homomorphism  $\pi$  factors through the Levi subgroup  $L = {}^-P/{}^-P_u$ . Put  $T_0 = \pi(T)$ . Then  $\pi$  induces an inclusion  $\pi^* : \Xi(T_0) \hookrightarrow \Xi(T)$ . We call  $\pi$  *central* if  $\pi^*(S(G_0)) \subseteq S(L)$ . If  $\pi$  is smooth then it is central. Thus, in characteristic zero,  $\pi$  is always central. We may always choose  $\pi$  to be central, if we so wish: just replace  $G_0$  by  $\overline{G}_0 := {}^-P/(\ker \pi)^{\text{red}}$ .

The following is well-known in characteristic zero:

**2.1. Lemma.** *Let  $X$  be the  $G$ -variety induced from the  $G_0$ -variety  $X_0$ .*

*a)  $X$  is spherical if and only if  $X_0$  is spherical.*

*Assume this. Then:*

*b)  $\Xi(X) = \Xi(X_0)$ . In particular,  $\text{rk } X = \text{rk } X_0$ .*

*c)  $\mathcal{V}(X) = \mathcal{V}(X_0)$ .*

*Assume, additionally, that  $\pi$  is central. Then*

*d)  $\Sigma(X) = \Sigma(X_0)$ .*

*e)  $S^{(p)}(X) = S^{(p)}(X_0) \cup S(\ker \pi)$ .*

*Proof.* Assertions [a\)](#) and [b\)](#) follow from the fact that  $X$  contains  $P \times^L X_0 = P_u \times X_0$  as a  $B$ -invariant open subset.

Up to a positive scalar, the elements of  $\mathcal{V}(X)$  correspond to smooth embeddings  $X \hookrightarrow \overline{X}$  such that  $D = \overline{X} \setminus X$  is a homogeneous divisor. The canonical morphism  $X \rightarrow G/{}^-P$  extends to  $\overline{X}$ . Thus,  $\overline{X}$  is of the form  $G \times^{} {}^-P \overline{X}_0$  where  $\overline{X}_0 \setminus X_0$  is a homogeneous divisor, as well. Hence it corresponds to an element of  $\mathcal{V}(X_0)$ . This easily implies [c\)](#). Assertion [d\)](#) is now a direct consequence of [c\)](#) and the definition of spherical roots.

Finally, assume  $P_\alpha$ ,  $\alpha \in S(G)$  stabilizes the open  $B$ -orbit  $X_1$  in  $X$ . Then its image  $P_u$  in  $G/{}^-P$  is stabilized, as well. This shows  $\alpha \in S(L)$ . In that case,  $P_\alpha$  stabilizes  $X_1$  if and only if  $\pi(P_\alpha) \subseteq G_0$  stabilizes the open  $B_0$ -orbit in  $X_0$ . Assertion [e\)](#) follows.  $\square$

Of particular importance is the case when  $X_0 = G_0/H_0$  is homogeneous. Then  $X = G/H$  is homogeneous, as well, with  $H = \pi^{-1}(H_0)$ . If  $H$  cannot be obtained by induction in a non-trivial way, i.e., with  $\dim G_0 < \dim G$ , then it is called *cuspidal*. Thus, this means two things:

*a) The only parabolic  $P \subseteq G$  with  $P_u \subseteq H \subseteq P$  is  $P = G$ .*

*b) The only connected normal subgroup  $K$  of  $G$  with  $K \subseteq H$  is  $K = 1$ .*

Observe that cuspidality is preserved under isogenies. More precisely, image and preimage of a cuspidal subgroup under an isogeny are cuspidal.

Any finite subgroup  $H$  of the 1-dimensional torus  $G = \mathbf{G}_m$  is certainly cuspidal. Spherical varieties of rank which are parabolically induced from a torus are called *horospherical*. In other words, a spherical subgroup  $H \subset G$  of corank 1 is horospherical if it is of the

form  $\ker \chi$  where  $\chi$  is a non-trivial character of a parabolic subgroup of  $G$ . We are going to prove later:

**2.2. Lemma.** *Assume that  $G$  contains a cuspidal spherical subgroup of corank 1. Then either  $G \cong \mathbf{G}_m$  or  $G$  is semisimple.*

Hence, it suffices to treat the case that  $G$  is semisimple. In the following it is convenient to assume that  $G$  is even of adjoint type which is obviously not a big loss of generality. The main classification theorem is now:

**2.3. Theorem.** *Let  $G$  be a semisimple group of adjoint type and let  $H \subset G$  be a cuspidal spherical subgroup of corank 1. Then:*

- a) *The pair  $(G, H)$  appears in the table §7.*
- b) *The coefficients of the spherical root  $\sigma \in \Sigma(G/H)$  are indicated in the column with caption “ $\sigma$ ”. The set  $S^{(p)}(G/H)$  is denoted in the form of black dots.*

**Remark.** The table is a bit condensed in the sense that an entry  $\langle s \rangle \cdot H_0$  denotes the two groups  $H = H_0$  and its normalizer  $H = \langle s \rangle H_0$ . In characteristic  $p \neq 2$ , the spherical roots for these groups differ by a factor of 2 which is also indicated.

Before we proceed to the proof of [Theorem 2.3](#) we discuss, following Luna, the notion of an “abstract spherical root” of  $G$ :

**2.4. Definition.** a) A spherical root of  $G$  is an element  $\sigma \in \mathbb{Z}S$  such that there is a spherical  $G$ -variety  $X$  of rank 1 with  $\Sigma(X) = \{\sigma\}$ . The set of spherical roots of  $G$  is denoted by  $\Sigma(G)$ .

- b) A spherical root  $\sigma$  of  $G$  is *compatible* to  $S' \subseteq S$  if there is a spherical  $G$ -variety  $X$  of rank 1 with  $\Sigma(X) = \{\sigma\}$  and  $S^{(p)}(X) = S'$ .

We introduce the *support*  $|\sigma| \subseteq S$  of a spherical root  $\sigma$  as the smallest set of simple roots needed to express it. In other words,

$$(2.1) \quad |\sigma| := \{\alpha \in S \mid n_\alpha \neq 0\}$$

when

$$(2.2) \quad \sigma = \sum_{\alpha \in S} n_\alpha \alpha \quad \text{with } n_\alpha = n_\alpha(\sigma) \in \mathbb{Z}_{\geq 0}.$$

Now, the following recipe on how to determine all spherical roots and all compatible sets follows directly from [Lemma 2.1](#).

- a)  $\sigma \in \Sigma(G)$  if and only if  $(|\sigma|, \sigma)$  appears in table §7.
- b)  $\sigma$  and  $S' \subseteq S$  are compatible if and only if all  $\alpha \in S' \setminus |\sigma|$  are orthogonal to  $\sigma$  and  $|\sigma| \cap S'$  consists of exactly the “black vertices” in the diagram of  $\sigma$  in table §7.

**Example.** The spherical roots for the root system  $A_3$  are:

$$\begin{aligned} &\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \\ &2\alpha_1, 2\alpha_2, 2\alpha_3 \text{ for } p \neq 2 \\ &\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_3 \\ &\alpha_1 + q\alpha_3, q\alpha_1 + \alpha_3 \text{ for } q = p^a > 1 \end{aligned}$$

The roots in the last row are new to positive characteristic. In particular, as opposed to characteristic 0, a fixed group may have infinitely many abstract spherical roots.

### 3. Proof of the classification

This section is devoted to the proof of [Theorem 2.3](#). Let  $X = G/H$  be spherical of rank 1. Then  $N_{\mathbb{Q}}(X) \cong \mathbb{Q}$  and  $\mathcal{V}(X)$  is either equal to  $N_{\mathbb{Q}}(X)$  (in case  $X$  is horospherical) or a halfline (otherwise). Thus, it follows from the theory of spherical embeddings (see [\[Kno91\]](#)) that  $X$  admits a unique smooth equivariant completion  $\overline{X}$  such that  $\partial X := \overline{X} \setminus X$  is pure of codimension one and consists of either two (horospherical case) or one (otherwise) homogeneous components. The embedding  $\overline{X}$  is maximal in the following sense: Let  $X'$  be a complete  $G$ -variety and  $\varphi : X' \rightarrow \overline{X}$  a birational  $G$ -equivariant morphism. Then  $\varphi$  is an isomorphism.

**3.1. Lemma.** *Let  $H \subset G$  be a spherical subgroup of corank 1 and let  $H \subseteq K \subseteq G$ . Then either  $K/H$  or  $G/K$  is a complete variety.*

*Proof* (same as [\[Bri89, 1.3 Lemme\]](#)). Assume  $Y := K/H$  is not a complete variety. Then it is not closed in  $\overline{X}$ . Let  $\overline{Y}$  denote its closure. Then  $G \times^K \overline{Y} \rightarrow \overline{X}$  is an isomorphism, by maximality. Thus, by inverting this morphism, we obtain an equivariant morphism  $\overline{X} \rightarrow G/K$  which implies that  $G/K$  is a complete variety.  $\square$

A first application is the

*Proof of [Lemma 2.2](#).* Let  $H \subset G$  be spherical of corank 1, let  $Z \subseteq G$  be the connected center of  $G$ , and put  $K := ZH$ . Then  $H$  is normal in  $K$  with  $K/H = Z/(Z \cap H)$  a torus. Thus, if  $G/K$  is complete then  $K$  is a parabolic in  $G$  and  $H$  is induced from  $K/H$ . Cuspidality implies  $G = K/H \cong \mathbf{G}_m$ . If, on the other hand,  $K/H$  is complete then  $H = K$ , i.e.,  $Z \subseteq H$ . But then  $Z = 1$  since  $H$  is cuspidal, i.e.,  $G$  is semisimple.  $\square$

From here to the end of this section,  $G$  will denote a semisimple group of adjoint type (unless stated otherwise). Moreover,  $H \subset G$  is spherical of corank 1. The classification forks then into two cases:

**3.2. Lemma.** *Let  $H \subset G$  be cuspidal. Then  $H$  is reductive or there is a parabolic subgroup  $P \subset G$  with  $P = HP_u$ .*

*Proof* (similar to [\[Bri89, 1.3 Théorème\]](#)). By a theorem of Borel-Tits [\[BT71, 3.1\]](#), there is a parabolic subgroup  $P \subseteq G$  with  $H \subseteq P$  and  $H_u \subseteq P_u$ . Choose  $P$  minimal with this property and put  $K = HP_u$ . Then either  $G/K$  or  $K/H$  is complete.

In the first case,  $K$  is parabolic in  $G$ . Since  $H \subseteq K \subseteq P$  and  $H_u \subseteq H_u P_u = K_u$  we get  $K = P$  by minimality of  $P$ .

In the second case,  $K/H = P_u/(H \cap P_u)$  is affine, connected, and complete, hence trivial. This and cuspidality imply  $P = G$  and  $H_u = P_u = 1$ , i.e.,  $H$  is reductive.  $\square$

Before we proceed we transfer a result of Dynkin [\[Dyn52, 15.1 Theorem\]](#) for semisimple Lie algebras in characteristic 0 to semisimple groups in arbitrary characteristic. The proof is basically the same.

**3.3. Lemma.** *Let  $G$  be a semisimple group with  $G_1, \dots, G_s$  its simple normal subgroups. Let  $H \subset G$  be maximal among connected reductive proper subgroups. Then either*

- a)  $H = H_i \prod_{j \neq i} G_j$  for some  $i$  where  $H_i \subset G_i$  is a maximal connected reductive proper subgroup or
- b)  $H = H_{ij} \prod_{l \neq i, j} G_l$  for some  $i \neq j$  where  $H_{ij} \subset G_i G_j$  is a connected subgroup such that the projections  $H_{ij} \rightarrow G_i / (G_i \cap G_j)$  and  $H_{ij} \rightarrow G_j / (G_i \cap G_j)$  are isogenies.

*Proof.* Without loss of generality we may assume  $G = G_1 \times \dots \times G_s$  with  $s \geq 2$ . Suppose that one of the projections  $H \rightarrow G_i$  is not surjective and denote its image by  $H_i$ . Then  $H$  is contained in, hence is equal to  $H_i \prod_{j \neq i} G_j$ .

Now assume that all projections  $H \rightarrow G_i$  are surjective. Then for every  $i$  there is a simple normal subgroup  $N \subseteq H$  such that the composition  $N \hookrightarrow H \hookrightarrow G \twoheadrightarrow G_i$  is an isogeny. All other factors of  $H$  are then mapped to the trivial group in  $G_i$ . Furthermore, there must be one factor  $N$  of  $H$  which is being mapped surjectively to more than one factor, say  $G_i$  and  $G_j$ , of  $G$  since otherwise  $H = G$ . Put  $H_{ij} = N$ . Thus,  $H$  is contained in, hence is equal to  $H_{ij} \prod_{l \neq i, j} G_l$ .  $\square$

**3.4. Lemma.** *Assume  $G$  is not simple and that  $H$  is cuspidal and reductive. Then  $G/H$  is isomorphic to  $(\mathrm{PGL}(2) \times \mathrm{PGL}(2)) / (\mathrm{id} \times F_q) \mathrm{PGL}(2)$  where  $F_q$  is a Frobenius morphism.*

*Proof* (similar to [Bri89, 2.2 Corollaire]). Assume first that  $H$  is connected. Then  $H$  is maximal among all connected reductive subgroups of  $G$  (Lemma 3.1). Since  $H$ , being cuspidal, does not contain any simple factor of  $G$ , Lemma 3.3 implies that  $G$ , being of adjoint type, is the direct product of two simple factors,  $G = G_1 \times G_2$ , and that  $\varphi_i : H \rightarrow G_i$  is an isogeny. Thus, there is a finite morphism  $H \times H / \Delta(H) \rightarrow G/H$  which implies that  $H$ , as an  $H \times H$ -variety, is of rank 1. But this rank equals the rank of  $H$  as a group. We conclude that the  $G_i$  are semisimple, of rank one, and of adjoint type and therefore isomorphic to  $\mathrm{PGL}(2)$ . Any isogeny from  $\mathrm{SL}(2)$  to  $\mathrm{PGL}(2)$  contains the (schematic) center of  $\mathrm{SL}(2)$  in its kernel. Thus, since we  $\varphi_1 \times \varphi_2$  is an embedding we see that also  $H \cong \mathrm{PGL}(2)$ , as well. Moreover,  $\varphi_1$  and  $\varphi_2$  must be (conjugate to) powers of the Frobenius morphism and one of them is the identity. Thus, up to a switch of factors, we have  $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$  and  $H = (\mathrm{id} \times F_q) \mathrm{PGL}(2)$ .

Finally, let  $H$  be not necessarily connected. Then  $H^0 = (\mathrm{id} \times F_q) \mathrm{PGL}(2)$  and  $H^0 \subseteq H \subseteq N_G(H^0) = H^0$ , i.e.,  $H = H^0$ .  $\square$

**3.5. Lemma.** *Assume that  $G$  is simple and  $H$  is cuspidal and reductive. Then  $(G, H)$  appears in the table §7.*

*Proof.* Here we refrain from generalizing the arguments of Brion in [Bri89, 2.3, 2.4]. Instead, with [KR13] we have now a classification of *all* connected spherical reductive subgroups of simple groups at our disposal. This generalizes Krämer's classification [Krä79] in characteristic zero. In a nutshell, the outcome of [KR13] is that, up to an isogeny of  $G$ , there appears only one more case in positive characteristic, namely  $H = \mathrm{G}_2 \times \mathrm{SL}(2) \subset G = \mathrm{Sp}(8)$  in char  $k = 2$ . But that case has rank 3. As a result, all pairs  $H \subset G$  appear in Krämer's list and therefore must appear in Akhiezer's list [Akh83], as well (always up to an isogeny of  $G$ ). The non-connected cases are easily determined by computing normalizers.  $\square$

According to Lemma 3.2, the last remaining step in the classification is:



**3.6. Lemma.** *Assume that  $H$  is cuspidal and that there is a parabolic subgroup  $P \subset G$  with  $P = HP_u$ . Then  $(G, H)$  is either isomorphic to  $(\mathrm{PGL}(n), \mathrm{GL}(n-1))$ ,  $n \geq 2$  or one of the non-reductive pairs in the table §7.*

*Proof* (generalization of [Bri89, 2.1]). Assume first that  $H$  is connected. Let  $L \subseteq P$  be a Levi subgroup. Then, by assumption, the homomorphism  $H \rightarrow P/P_u \cong L$  is surjective. Thus,  $H_u$  is mapped to a unipotent normal subgroup of  $L$ , i.e., to 1 which means  $H_u \subseteq P_u$ . Let  $Z \subseteq L$  be the center of  $L$ . It is a torus since  $G$  is of adjoint type. Hence, it can be lifted to a subtorus  $Z'$  of  $H$ . Because  $Z'$  is  $P$ -conjugate to  $Z$  we may, without loss of generality, assume that  $Z = Z' \subseteq H$ . But then  $L = C_P(Z) \subseteq H$  (see, e.g., [Bor91, 11.14 Cor. 2]), i.e.,  $H = L \ltimes H_u$ .

Recall the maximal compactification  $\overline{X}$  of  $X = G/H$  and let  $\overline{Y} \subseteq \overline{X}$  be the closure of  $Y = P/H \subseteq G/H$ . Then  $G \times^P \overline{Y} \rightarrow \overline{X}$  is proper and birational, hence an isomorphism by the maximality property of  $\overline{X}$ . This shows that  $\overline{Y}$  consists of two  $P$ -orbits, namely  $Y$  and  $\partial Y = \overline{Y} \setminus Y$ . The latter is a complete  $P$ -orbit of codimension 1 in  $\overline{Y}$ . In particular, since  $P_u$ , being solvable, has a fixed point in  $\partial Y$  it acts in fact trivially and  $L$  acts transitively on  $\partial Y$ .

The 1-dimensional root subgroups  $U_\alpha \subset B$ ,  $\alpha \in S$ , generate  $U = (B, B)$ . Thus, since  $H$  is not horospherical, there is a simple root  $\alpha$  such that  $U_\alpha \not\subseteq H$ . This means  $U_\alpha \subseteq P_u$  but  $U_\alpha \not\subseteq H_u$ . Then  $C := U_\alpha H/H \subseteq Y$ , the  $U_\alpha$ -orbit in  $P/H$ , is an affine curve. It is closed, since  $Y = P_u/H_u$  is affine and  $U_\alpha$  is unipotent. Now let  $\overline{C}$  be its closure in  $\overline{Y}$ . Let  ${}^{-}B \subseteq G$  be the Borel subgroup opposite to  $B$ . Then  $U_\alpha$  is normalized by  ${}^{-}B \cap L$  which implies that  $\overline{C}$  is  ${}^{-}B \cap L$ -invariant. Because  ${}^{-}B \cap L$  is a Borel subgroup of  $L$ , we infer that  $L\overline{C}$  is an irreducible closed  $L$ -invariant subset of  $\overline{Y}$ . Since  $\overline{C}$  meets  $\partial Y$  and  $L$  acts transitively on  $\partial Y$  this implies that  $\partial Y \subseteq L\overline{C}$ . For dimension reasons we get  $\overline{Y} = L\overline{C}$  and therefore  $Y = LC$ . But the center  $Z \subseteq L$  has only two orbits in  $C$ , namely  $\{0\}$  and its complement. This shows that  $L$  acts transitively on  $Y \setminus \{0\}$ .

At this point we can show that it was no loss of generality to assume that  $H$  is connected. Indeed let  $\tilde{H} \subseteq P$  be a subgroup with  $\tilde{H}^0 = H$ . Then  $\tilde{H}/H$  is a finite  $L$ -invariant subset of  $Y$ . Hence  $\tilde{H}/H = \{0\}$  and therefore  $\tilde{H} = H$ .

Let  $Q \subseteq G$  be the parabolic generated by  $P$  and  $U_{-\alpha}$ . Then  $U_\alpha \not\subseteq Q_u$  implies that  $\alpha$  is not a weight in  $Q_u H_u$ . Hence  $Q_u H_u/H_u$  is a proper  $L$ -invariant subvariety of  $Y = P_u/H_u$  which means  $Q_u \subseteq H_u \subseteq H \subseteq Q$ . Since  $H$  is cuspidal we conclude that  $Q = G$ , i.e., that  $P$  is a maximal parabolic subgroup corresponding to the set of simple roots  $\Sigma' = \Sigma(G) \setminus \{\alpha\}$ . Moreover, the cuspidality of  $H$  implies that  $G$  is simple and that  $\Sigma(G)$  is connected.

Since  $P_u$  acts transitively on  $Y$  and  $L$  acts transitively on  $Y \setminus \{0\}$  the action of  $P$  on  $Y$  is doubly transitive. According to [Kno83, Satz 2] there is an isomorphism  $Y \cong \mathbf{A}^n$  such that the  $P$ -action is given by a surjective homomorphism  $\pi : P \twoheadrightarrow (\mathbf{G}_m \cdot G_0) \ltimes \mathbf{G}_a^n$  where  $\mathbf{G}_a^n$  acts by translations,  $\mathbf{G}_m$  acts by scalars, and  $G_0$  is either  $\mathrm{SL}(n)$ ,  $n \geq 2$  or  $\mathrm{Sp}(n)$ ,  $n \geq 2$  even or  $\mathbf{G}_2$ , with  $n = 6$  and  $\mathrm{char} k = 2$ . This means in particular that  $Y = P_u/H_u$  is a linear representation of  $L$ . It is irreducible with lowest weight  $\alpha$ .

We conclude that  $\alpha$  is an “end” of  $\Sigma$  and  $\Sigma'$  is of type A or C in general or of type B if  $\mathrm{char} k = 2$ . Moreover,  $-\alpha$  is a  $p$ -power multiple of the canonical representation. Up to

isomorphism, this leaves precisely the following cases:

$$(3.1) \quad (\Sigma, \alpha) = (A_n, \alpha_1), (B_n, \alpha_n), (C_n, \alpha_1), (G_2, \alpha_1)$$

in arbitrary characteristic and

$$(3.2) \quad (\Sigma, \alpha, p) = (C_n, \alpha_n, 2), (B_n, \alpha_1, 2), (G_2, \alpha_2, 3)$$

in characteristic  $p > 0$ . Using a non-central isogeny, the latter cases reduce to the former.

Clearly all cases appear (see table). Moreover,  $H$  contains the maximal torus  $T$ . Therefore,  $H_u$  is generated by all  $U_\beta$  where  $\beta$  is a root of  $P_u$  which is not a weight of  $Y$ . This shows uniqueness of  $H$ .  $\square$

Finally, we prove part *b*) of [Theorem 2.3](#). For this we use the following

**3.7. Lemma.** *Let  $X = G/H$  be defined over a field of characteristic  $p > 0$  and that  $X$  can be lifted to a homogeneous variety  $X_0$  over a field of characteristic 0. Then  $X$  is spherical if and only if  $X_0$  is. Moreover, there are equalities  $\Xi_p(X) = \Xi_p(X_0)$  and  $S^{(p)}(X) = S^{(p)}(X_0)$ .*

*Proof.* By assumption, there is a complete discrete valuation ring  $R$  with residue field  $k$  and uniformizer  $\pi \in R$  and a smooth  $R$ -scheme  $\mathcal{X}$  which has  $X$  and  $X_0$  as special fiber and generic geometric fiber, respectively. Then the equivalence of  $X$  and  $X_0$  being spherical is [\[KR13, Thm. 3.4\]](#).

According to [\[KR13, Lem. 3.1\]](#), for every  $B$ -seminvariant  $f$  on  $X$  there is an  $n \geq 1$  such that  $f^n$  extends to a semiinvariant on  $\mathcal{X}$ . Moreover, [\[FvdK10, Prop. 41\]](#), the exponent  $n$  can be chosen to be a power of  $p$ . This combined shows  $\Xi(X) \subseteq \Xi_p(X_0)$ .

Conversely, for let  $\chi \in \Xi(X_0)$  let  $f$  be a semiinvariant on  $X_0$  with character  $\chi_f = \chi$ . After possibly replacing  $R$  by a (ramified) extension we may assume that  $f$  is a rational function on  $\mathcal{X}$  which has poles at most along  $X$ . Thus, there is a unique exponent  $m \in \mathbb{Z}$  such that  $\tilde{f} := \pi^m f$  is regular on  $\mathcal{X}$ . Then, the restriction of  $\tilde{f}$  to  $X$  is a  $B$ -semiinvariant with character  $\chi$ . This even shows  $\Xi(X_0) \subseteq \Xi(X)$ .

For  $\mathcal{X}$  quasiaffine, the equality  $S^{(p)}(X) = S^{(p)}(X_0)$  follows from  $S^{(p)}(X) = \{\alpha \in S \mid \alpha^r = 0\}$ . The general case is reduced to this by passing to an affine cone over  $X$ .  $\square$

Now let  $H \subset G$  be a member of [Table §7](#) which can be lifted to characteristic zero. Then  $\Xi(X_0)$  and  $S^{(p)}(X_0)$  are well known (see, e.g., [\[Was96, Table 1\]](#)). From this and the Lemma it is easy to determine  $\Xi_p(X)$  and  $S^{(p)}(X)$ .

All other cases are isogenous to liftable ones. So the corresponding data can be easily calculated, as well. This finishes the proof of [Theorem 2.3](#).

#### 4. The structure of the valuation cone

Our main application for classifying spherical roots is to extend Brion's Theorem [\[Bri90\]](#) on the structure of the valuation cone of a spherical variety.

In the following, we choose an auxiliary  $W$ -invariant rational scalar product on  $\Xi_{\mathbb{Q}}(T)$ . For any  $0 \neq \sigma \in \Xi_{\mathbb{Q}}(X)$  let

$$(4.1) \quad s_\sigma(\chi) = \chi - \sigma^r(\chi)\sigma, \quad \text{with } \sigma^r(\chi) := 2 \frac{(\chi, \sigma)}{(\sigma, \sigma)}$$



the unique orthogonal reflection of  $\Xi_{\mathbb{Q}}(X)$  with  $s_{\sigma}(\sigma) = -\sigma$ .

**4.1. Lemma.** *Let  $X$  be a spherical  $G$ -variety and  $\sigma \in \Sigma(X)$ . Then:*

- a) There is  $n_{\sigma} \in W$  with  $n_{\sigma}|_{\Xi_{\mathbb{Q}}(X)} = s_{\sigma}$ .*
- b) Assume  $\sigma \notin 2S$ . Then there is a root  $\beta$  of  $G$  with  $\sigma^r = \beta^r$ .*

*Proof.* We check all items of the table §7.

1. case:  $\sigma = u\alpha$  where  $\alpha$  is a root of  $G$  and  $u \in \{1, 2\}$ . Then clearly  $n_{\sigma} = s_{\alpha}$  works for *a*). Moreover, *b*) is trivial for  $u = 1$ . So let  $u = 2$ . Then  $p \neq 2$  and there are three cases to consider:

- $|\sigma| = B_n$  ( $n \geq 2$ ),  $\alpha = \alpha_1 + \dots + \alpha_n$ , and  $\alpha_n \in S^{(p)}(X)$ . Then  $\alpha_n^r = 0$ . On the other hand  $\sigma^r = \frac{1}{2}\alpha^r = \frac{1}{2}(2\beta^r + \alpha_n^r) = \beta^r$  where  $\beta = \alpha_1 + \dots + \alpha_{n-1}$  is a root.
- $|\sigma| = G_2$ ,  $\alpha = 2\alpha_1 + \alpha_2$ , and  $\alpha_2 \in S^{(p)}$ . Then  $\sigma^r = \frac{1}{2}\alpha^r = \frac{1}{2}(2\alpha_1^r + 3\alpha_2^r) = \alpha_1^r$ .
- $|\sigma| = G_2$ ,  $\alpha = 3\alpha_1 + 2\alpha_2$ ,  $\alpha_1 \in S^{(p)}$ , and  $p = 3$ . Then  $\sigma^r = \frac{1}{2}\alpha^r = \frac{1}{2}(\alpha_1^r + 2\alpha_2^r) = \alpha_2^r$ .

For the other cases we first prove *a*). For this, we use the following observation: let  $\sigma = u\alpha + v\beta$  with  $u, v \in \mathbb{Q}_{>0}$  and  $\alpha, \beta$  orthogonal roots of  $G$ . Assume  $u^{-1}\alpha^r - v^{-1}\beta^r = 0$ . Then  $s_{\sigma} = s_{\alpha}s_{\beta}|_{\Xi_{\mathbb{Q}}(X)}$ . Indeed, the assumptions imply  $\sigma^r = u^{-1}\alpha^r = v^{-1}\beta^r$ . Hence

$$(4.2) \quad s_{\alpha}s_{\beta}(\chi) = \chi - \alpha^r(\chi)\alpha - \beta^r(\chi)\beta = \chi - \sigma^r(\chi)\sigma = s_{\sigma}(\chi)$$

for all  $\chi \in \Xi_{\mathbb{Q}}(X)$ .

2. case:  $\sigma = \alpha_1 + q\alpha_2$  with  $\alpha_1, \alpha_2 \in S$  orthogonal. Then  $\alpha_1^r - q^{-1}\alpha_2^r = 0$  by [Kno13, (5.2)].

3. case: In the remaining four cases, we claim that there is a decomposition  $\sigma = u\alpha + v\beta$  where  $\alpha, \beta$  are orthogonal roots of  $G$  and  $u^{-1}\alpha^{\vee} - v^{-1}\beta^{\vee} \in \langle \gamma^{\vee} \mid \gamma \in S^{(p)} \rangle_{\mathbb{Q}}$ . Indeed:

$ \sigma  = A_3$	$\sigma = \alpha_1 + 2\alpha_2 + \alpha_3 = \alpha + \beta$ $\alpha = \alpha_1 + \alpha_2$ $\beta = \alpha_2 + \alpha_3$ $\alpha^{\vee} - \beta^{\vee} = \alpha_1^{\vee} - \alpha_3^{\vee}$	$= \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4,$ $= \varepsilon_1 - \varepsilon_3,$ $= \varepsilon_2 - \varepsilon_4,$ $= (\varepsilon_1 - \varepsilon_2) - (\varepsilon_3 - \varepsilon_4)$
$ \sigma  = B_3$	$\sigma = \alpha_1 + 2\alpha_2 + 3\alpha_3 = \alpha + \beta$ $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3$ $\beta = \alpha_2 + \alpha_3$ $\alpha^{\vee} - \beta^{\vee} = \alpha_1^{\vee} - \alpha_2^{\vee}$	$= \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$ $= \varepsilon_1 + \varepsilon_3,$ $= \varepsilon_2,$ $= (\varepsilon_1 - \varepsilon_2) - (\varepsilon_2 - \varepsilon_3)$
$ \sigma  = D_n$	$\sigma = 2\alpha_1 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n = \alpha + \beta$ $\alpha = \alpha_1 + \dots + \alpha_{n-2} + \alpha_{n-1}$ $\beta = \alpha_1 + \dots + \alpha_{n-2} + \alpha_n$ $\alpha^{\vee} - \beta^{\vee} = \alpha_{n-1}^{\vee} - \alpha_n^{\vee}$	$= 2\varepsilon_1,$ $= \varepsilon_1 - \varepsilon_n,$ $= \varepsilon_1 + \varepsilon_n,$ $= (\varepsilon_{n-1} - \varepsilon_n) - (\varepsilon_{n-1} + \varepsilon_n)$
$ \sigma  = C_3$	$\sigma = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 = 2\alpha + \beta$	$= 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$
$(p = 2)$	$\alpha = \alpha_1 + \alpha_2 + \alpha_3$ $\beta = 2\alpha_2 + \alpha_3$ $\frac{1}{2}\alpha^{\vee} - \beta^{\vee} = \frac{1}{2}\alpha_1^{\vee} - \frac{1}{2}\alpha_2^{\vee}$	$= \varepsilon_1 + \varepsilon_3,$ $= 2\varepsilon_2,$ $= \frac{1}{2}(\varepsilon_1 - \varepsilon_2) - \frac{1}{2}(\varepsilon_2 - \varepsilon_3).$

This shows *a*) in all cases. For *b*) use  $\sigma^r = u^{-1}\alpha^r = v^{-1}\beta^r$  and the fact that in each case one of  $u$  or  $v$  is 1. □

**Remarks.** a) The proof shows that  $n_\sigma$  can be chosen to be either a simple reflection  $s_\alpha$  or  $n_\sigma = s_\alpha s_\beta$  where  $\alpha$  and  $\beta$  are *very orthogonal roots* meaning that there is  $w \in W$  such that  $w\alpha$  and  $w\beta$  are orthogonal simple roots.

b) The element  $n_\sigma$  can be chosen independently of the choice of the scalar product on  $\Xi(T)$ . Hence, also  $s_\sigma$  is independent of the scalar product.

**4.2. Definition.** The subgroup of  $W_X \subseteq \text{GL}(\Xi_{\mathbb{Q}}(X))$  generated by all  $s_\sigma$ ,  $\sigma \in \Sigma(X)$ , is called the *little Weyl group of  $X$* .

**4.3. Theorem.** *Let  $X$  be a spherical variety. Then:*

- a) *The little Weyl group  $W_X$  is finite.*
- b) *The groups  $\Xi_p(X)$  and  $\mathbb{Z}S \cap \Xi_p(X)$  are  $W_X$ -invariant.*
- c) *The set  $R_X := W_X \Sigma(X)$  is a (finite) root system with Weyl group  $W_X$ .*

*Proof.* a) The little Weyl group is finite since, by [Lemma 4.1a](#)), it is a subquotient of  $W$ .

b) Let  $\sigma \in \Sigma(X)$ . Then  $\sigma \in \Xi_p(X)$ . Moreover, it follows from [\[Kno13, Corollary 2.4\]](#) (for  $\sigma \in 2S$ ) and [Lemma 4.1b](#)) (for  $\sigma \notin 2S$ ) that  $\sigma^r$  takes values in  $\mathbb{Z}_p$  on  $\Xi_p(X)$ . This combined implies the  $s_\sigma$ -invariance of  $\Xi_p(X)$ . Then the invariance of  $\mathbb{Z}S \cap \Xi_p(X)$  follows from the  $W$ -invariance of  $\mathbb{Z}S$  and [Lemma 4.1a](#)).

c) follows from the fact that  $R_X$  consists of primitive vectors in the  $W_X$ -invariant lattice  $\mathbb{Z}S \cap \Xi_p(X)$ .  $\square$

**Remark.** The group  $\Xi(X)$  itself is, in general, not  $W_X$ -invariant. Take for example  $G = \text{GL}(2)$  and  $X = \mathbf{A}^2 \times {}^{(q)}\mathbf{P}^1$ . Here  ${}^{(q)}\mathbf{P}^1$  denotes the projective line with the Frobenius twisted  $G$ -action. Then  $\Xi(X) = \mathbb{Z}q\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$  which is not  $s_\alpha$ -stable unless  $q = 1$ .

**4.4. Corollary.** *The valuation cone  $\mathcal{V}(X)$  of a spherical variety  $X$  is the union of Weyl chambers for the little Weyl group  $W_X$ .*

**Remark.** Schalke, [\[Sch11\]](#), has shown that if  $p = 2$  and  $X = \text{PGL}(3)/\text{SO}(3)$  then  $\Sigma(X) = \{\alpha_1, \alpha_1 + \alpha_2\}$ . Thus  $\mathcal{V}(X)$  can be identified with the set of rational triples  $(x_1, x_2, x_3)$  with  $x_1 + x_2 + x_3 = 0$  and  $x_1 \leq x_2, x_3$  which is the union of the two chambers  $\{x_1 \leq x_2 \leq x_3\}$  and  $\{x_1 \leq x_3 \leq x_2\}$ .

The next goal is now to prove that  $\mathcal{V}(X)$  is actually just one Weyl chamber, provided that  $p \neq 2$ . A constraint on the angles between spherical roots is given by the following theorem. Its proof is deferred to [section 5](#).

**4.5. Theorem.** *Let  $X$  be a spherical variety and let  $\sigma, \tau \in \Sigma(X)$  be distinct spherical roots with  $(\sigma, \tau) > 0$ . Then  $p = 2$  and, up to a switch of  $\sigma$  and  $\tau$ , the triple  $|\sigma| \cup |\tau|$ ,  $\sigma$ ,  $\tau$  is contained in the following table.*

	$ \sigma  \cup  \tau $	$\sigma$	$\tau$
	$\mathbf{A}_2$	$\alpha_1$	$\alpha_1 + \alpha_2$
(4.3)	$\mathbf{B}_n, n \geq 2$	$\alpha_2 + \dots + \alpha_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n$
	$\mathbf{C}_n, n \geq 2$	$2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$
	$\mathbf{G}_2$	$\alpha_2$	$\alpha_1 + \alpha_2$

**Remark.** In characteristic 2, all exceptional cases in [Theorem 4.5](#) do occur. Namely:

- $\Sigma(X) = \{\alpha_1, \alpha_1 + \alpha_2\} \subset \mathbf{A}_2$  where  $X = \text{SL}(3)/\text{SO}(3)$  (see [\[Sch11\]](#)).

- $\Sigma(X) = \{\alpha_2 + \dots + \alpha_n, \alpha_1 + 2\alpha_2 + \dots + 2\alpha_n\} \subset \mathbf{B}_n$ ,  $n \geq 2$  where  $X$  is the open  $\mathrm{SO}(2n+1)$ -orbit in the Grassmannian of codimension-2-subspaces in  $k^{2n+1}$  (the defining representation).
- $\Sigma(X) = \{2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n, \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n\} \subset \mathbf{C}_n$ ,  $n \geq 2$ , is isogenous to the preceding case, thus occurs as well. More concretely: the variety  $\mathrm{Sp}(2n)/\mathrm{O}(2n)$  is isomorphic to an affine space  $\mathbf{A}^{2n}$  on which  $\mathrm{Sp}(2n)$  acts by affine linear transformations. Now  $X$  is the open orbit in the set of affine lines of  $\mathbf{A}^{2n}$ .
- $\Sigma(X) = \{\alpha_2, \alpha_1 + \alpha_2\} \subset \mathbf{G}_2$  where  $X = G/H$  and  $H = \mathrm{SO}(3) \cdot U_{2\alpha_1+\alpha_2} U_{3\alpha_1+\alpha_2} U_{3\alpha_1+2\alpha_2} \subset P_{\alpha_2} \subset \mathbf{G}_2$ .

From [Theorem 4.5](#) we derive the main result of this paper:

**4.6. Theorem.** *Let  $X$  be a spherical variety which is defined over a field of characteristic  $p \neq 2$ . Then  $\Sigma(X)$  is a system of simple roots for the root system  $R_X = W_X \Sigma(X)$ .*

*Proof.* According to [Theorem 4.5](#), the angle between any two spherical roots is  $\geq \frac{\pi}{2}$ . Hence [\[Bou68, Ch. 5, §3, no. 5, Lemme 3\]](#) (or [\[Bri90, Théorème 3.1\]](#)) implies that  $\Sigma(X)$  is linearly independent. Now we argue as in the proof of [\[Bri90, Théorème 3.5\]](#).  $\square$

The following corollaries are immediate consequences:

**4.7. Corollary.** *Let  $p \neq 2$ . Then the valuation cone  $\mathcal{V}(X)$  of a spherical variety  $X$  is a Weyl chamber for the little Weyl group  $W_X$ .*

**4.8. Corollary.** *Let  $p \neq 2$  and  $X$  a spherical variety. Then the set  $\Sigma(X)$  of spherical roots is linearly independent. This means, in particular, that the valuation cone  $\mathcal{V}(X)$  is a cosimplicial cone.*

The theory of spherical embeddings, [\[Kno91\]](#), provides us with the following relaxed version of a wonderful embedding:

**4.9. Corollary.** *Let  $p \neq 2$  and let  $X = G/H$  be a homogeneous spherical variety such that  $H^{\mathrm{red}}$  is of finite index in its normalizer. Then there is an equivariant normal completion  $X \hookrightarrow \overline{X}$  with the following properties:*

- The boundary  $\overline{X} \setminus X$  is the union of  $r = \mathrm{rk} X$  irreducible divisors  $D_1, \dots, D_r$ .*
- The map  $I \mapsto (\bigcap_{i \in I} D_i)^{\mathrm{red}}$  is a bijection between subsets of  $\{1, \dots, r\}$  and the set of orbit closures in  $\overline{X}$ .*

*Proof.* The condition on the normalizer implies that the valuation cone  $\mathcal{V}(X)$  is pointed ([\[Kno91, Thm. 6.1\]](#)). Thus,  $\mathcal{C} = \mathcal{V}(X)$  defines a toroidal embedding  $X \hookrightarrow \overline{X}$ . Properties [a\)](#) and [b\)](#) follow from the fact that  $\mathcal{V}(X)$  is a simplicial cone.  $\square$

**Remark.** Observe, that, as opposed to wonderful varieties, we made no claims of smoothness or transversality.

[Theorem 4.5](#) has also the following consequence:

**4.10. Corollary.** *Every (internal) dihedral angle of the valuation cone equals either*

$$(4.4) \quad \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi, \frac{3}{4}\pi, \text{ or } \frac{5}{6}\pi.$$

*The last three values occur only for  $p = 2$ .*

## 5. The angles between spherical roots

This section is devoted to the proof of [Theorem 4.5](#). We are going to use the notation

$$(5.1) \quad |\sigma|_p := |\sigma| \cap S^{(p)}, \quad |\sigma|_s := |\sigma| \setminus S^{(p)}$$

which we call the *parabolic support* and the *singular support* of  $\sigma$ . An inspection of [table §7](#) shows that the singular support consists of either 1 or 2 elements.

We start with a trivial but useful observation, the saturation principle:

**5.1. Lemma.** *Let  $\sigma$  be a spherical root, let  $\alpha \in |\sigma|$ , and let  $\beta \in S^{(p)}$  be connected to  $\alpha$  in the Dynkin diagram (i.e.,  $(\alpha, \beta) < 0$ ). Then  $\beta \in |\sigma|$ , as well.*

*Proof.* Since  $\alpha$  and  $\beta$  are connected we have  $(\alpha, \beta) < 0$ . Now suppose  $\beta \notin |\sigma|$ . Then

$$(5.2) \quad 0 = (\beta, \sigma) = n_\alpha(\beta, \alpha) + \sum_{\gamma \in |\sigma| \setminus \{\alpha\}} n_\gamma(\beta, \gamma) < 0,$$

a contradiction. □

First we show that the angle between almost any two spherical roots is automatically obtuse, just for combinatorial reasons. The length of a root does not play a rôle for that. Therefore, we are considering only spherical roots  $\sigma$  which are *reduced*, i.e., for which  $\frac{1}{2}\sigma$  is not a spherical root.

**5.2. Lemma.** *Let  $\sigma$  and  $\tau$  be two distinct reduced spherical roots of  $G$  which are compatible with some subset  $S^{(p)} \subseteq S$ . Assume that both  $\sigma$  and  $\tau$  have connected support and that  $(\sigma, \tau) > 0$ . Then, up to a switch of  $\sigma$  and  $\tau$ , the triple  $|\sigma| \cup |\tau|, \sigma, \tau$  is contained in the following table:*

$ \sigma  \cup  \tau $	$\sigma$	$\tau$
$A_2$	$\alpha_1$	$\alpha_1 + \alpha_2$
$B_2$	$\alpha_1$	$\alpha_1 + \alpha_2$
$G_2$	$\alpha_2$	$\alpha_1 + \alpha_2$
<i>Additionally for <math>p = 2</math>:</i>		
$B_2$	$\alpha_1 + \alpha_2$	$\alpha_1 + 2\alpha_2$
$B_n, n \geq 2$	$\alpha_2 + \dots + \alpha_{n-1} + \alpha_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + 2\alpha_n$
$C_n, n \geq 3$	$2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$
<i>Additionally for <math>p = 3</math>:</i>		
$G_2$	$\alpha_1$	$3\alpha_1 + \alpha_2$

*Proof.* The case  $\#|\sigma| \cup |\tau| = 2$  can be easily handled case by case. These provide for six cases in the table. Therefore, assume from now on that  $\#|\sigma| \cup |\tau| > 2$ . Let  $\sigma = \sum_{\alpha \in S} n_\alpha \alpha$ . Then, because of

$$(5.4) \quad 0 < (\sigma, \tau) = \sum_{\alpha \in |\sigma|_s} n_\alpha(\alpha, \tau)$$

we have  $|\sigma|_s \cap |\tau|_s = |\sigma|_s \cap |\tau| \neq \emptyset$ .

First, we treat the case that  $\#|\sigma|_s = 1$ . Then  $|\sigma|_s \subseteq |\tau|$ . The saturation principle and the connectedness of  $|\sigma|$  imply that  $|\sigma| \subseteq |\tau|$ . If also  $\#|\tau|_s = 1$  then, by symmetry,  $|\tau| \subseteq |\sigma|$ , as well. An inspection of the list of spherical roots shows that this is not possible with  $\#|\sigma| > 2$ .

Now let  $|\tau|_s = 2$ . Then the saturation principle and the table leaves the following possibilities (recall  $\#|\tau| \geq 3$ ):

	$ \tau $	$\sigma$	$\tau$	$(\sigma, \tau)$
	$B_4$	$\alpha_2 + 2\alpha_3 + 3\alpha_4 = \varepsilon_2 + \varepsilon_3 + \varepsilon_4$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \varepsilon_1$	$= 0$
	$C_n, n \geq 2$	$\alpha_1 = \varepsilon_1 - \varepsilon_2$	$\alpha_1 + 2\alpha_2 + \dots + \alpha_n = \varepsilon_1 + \varepsilon_2$	$= 0$
	Additionally for $p = 2$ :			
(5.5)	$B_3$	$\alpha_2 + 2\alpha_3 = \varepsilon_2 + \varepsilon_3$	$\alpha_1 + \alpha_2 + \alpha_3 = \varepsilon_1$	$= 0$
	$B_n, n \geq 2$	$\alpha_1 = \varepsilon_1 - \varepsilon_2$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n = \varepsilon_1 + \varepsilon_2$	$= 0$
	$B_n, n \geq 2$	$\alpha_2 + \dots + \alpha_n = \varepsilon_2$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n = \varepsilon_1 + \varepsilon_2$	$> 0$
	$C_n, n \geq 2$	$2\alpha_2 + \dots + \alpha_n = 2\varepsilon_2$	$\alpha_1 + 2\alpha_2 + \dots + \alpha_n = \varepsilon_1 + \varepsilon_2$	$> 0$
	$C_3$	$\alpha_2 + \alpha_3 = \varepsilon_2 + \varepsilon_3$	$2\alpha_1 + 2\alpha_2 + \alpha_3 = 2\varepsilon_1$	$= 0$
	$C_4$	$2\alpha_2 + 4\alpha_3 + 3\alpha_4 = 2\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4$	$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = 2\varepsilon_1$	$= 0$

Thus, we obtain exactly the two series in the statement.

Finally, assume that  $\#|\sigma|_s = \#|\tau|_s = 2$ . Then  $|\sigma|_s = |\tau|_s$  would imply, as above, that  $|\sigma| = |\tau|$ , hence  $\sigma = \tau$ . This leaves the case  $|\sigma|_s = \{\alpha, \beta\}$ ,  $|\tau|_s = \{\beta, \gamma\}$ . We have

$$(5.6) \quad 0 < (\sigma, \tau) = n_\alpha(\alpha, \tau) + n_\beta(\beta, \tau)$$

which shows that  $(\beta, \tau) > 0$  and, by symmetry,  $(\beta, \sigma) > 0$ . If one goes through all possible pairs  $\beta, \sigma$ , the saturation principle shows that  $|\sigma| = \{\alpha\} \cup T$  and, by symmetry,  $|\tau| = \{\gamma\} \cup T$  where  $T = |\sigma| \cap |\tau|$ . Thus, we arrive at the following possibilities:

	$ \sigma  \cup  \tau $	$\sigma$	$\tau$
(5.7)	$D_n, n \geq 3$	$\alpha_1 + \dots + \alpha_{n-2} + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$	$\alpha_1 + \dots + \alpha_{n-2} + \alpha_n = \varepsilon_1 + \varepsilon_n$
	$B_3$	$\alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$	$\alpha_2 + \alpha_3 = \varepsilon_2$
	Additionally for $p = 2$ :		
	$C_3$	$\alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3$	$2\alpha_2 + \alpha_3 = 2\varepsilon_2$

In all these cases, we have  $(\sigma, \tau) = 0$ . □

*Proof of Theorem 4.5.* We treat first the case when the support of  $\sigma$  is not connected. Then  $\sigma = \alpha_1 + q\alpha_2$  with  $\alpha_1, \alpha_2 \in S$  orthogonal. Equation [Kno13, (5.2)] implies that  $\tau$  is orthogonal to  $\lambda := \alpha_1^\vee - q^{-1}\alpha_2^\vee$ . It follows that  $(\alpha_2, \tau) = x(\alpha_1, \tau)$  for some  $x > 0$ . Thus,  $0 < (\sigma, \tau) = y(\alpha_1, \tau)$  with  $y = 1 + q^{-1}x > 0$  which implies  $\alpha_1 \in |\tau|_s$ . By symmetry, also  $\alpha_2 \in |\tau|_s$ , hence  $\{\alpha_1, \alpha_2\} = |\tau|_s$ . Now one can use the table of spherical roots in §7 to show that this is only possible if  $\tau = \alpha_1 + q'\alpha_2$ . But then  $(\lambda, \tau) = 0$  implies  $q' = q$ , in contradiction to  $\sigma \neq \tau$ .

So, up to factors of 2, the pair  $\sigma, \tau$  is contained in Table (5.3). Looking at  $S^{(p)}$ , observe that  $\tau$  is necessarily reduced. Moreover, if  $p \neq 2$  then  $\sigma$  is reduced as well because of the parity condition [Kno13, Corollary 2.4].

To exclude other cases in (5.3) we use some results from [Kno13] which require that  $\sigma$  and  $\tau$  are *neighbors in*  $\Sigma(X)$ . This means that  $\mathbb{Q}_{\geq 0}\sigma + \mathbb{Q}_{\geq 0}\tau$  is a two-dimensional face of the cone  $\mathbb{Q}_{\geq 0}\Sigma(X)$ . Fortunately, this holds in all cases of interest:

**5.3. Lemma.** *Let  $\sigma, \tau \in \Sigma(X)$  be distinct spherical roots with  $|\sigma| \subseteq |\tau|$ . Then  $\sigma$  and  $\tau$  are neighbors in  $\Sigma(X)$ .*

*Proof.* Otherwise there is an equation

$$(5.8) \quad u\sigma + v\tau = t_1\eta_1 + \dots + t_s\eta_s$$

with  $u, v, t_1, \dots, t_s > 0$  and  $\eta_1, \dots, \eta_s \in \Sigma(X) \setminus \{\sigma, \tau\}$ . Clearly  $s \geq 1$  and  $|\eta_1| \subseteq |\sigma| \cup |\tau| = |\tau|$ . Because  $\eta_1$  is compatible to  $|\tau|_p$  and  $\#|\tau|_s \leq 2$  we infer that  $\eta_1$  is a linear combination of  $\sigma$  and  $\tau$ . But then one of  $\sigma, \tau$  or  $\eta_1$  could not be extremal in  $\mathbb{Q}_{\geq 0}\Sigma(X)$ .  $\square$

Next, we show that the three cases

$$(5.9) \quad (\mathbf{A}_2, \alpha_1, \alpha_1 + \alpha_2), (\mathbf{B}_2, \alpha_1, \alpha_1 + \alpha_2), (\mathbf{G}_2, \alpha_2, \alpha_1 + \alpha_2)$$

don't occur when  $p \neq 2$ . In all three cases, we have  $\sigma \in S^{(a)}$  and let  $D_1$  and  $D_2$  be the two colors moved by  $\sigma$ . Because of  $\delta_{D_1}^{(\sigma)} + \delta_{D_2}^{(\sigma)} = \sigma^r$ , we have  $\delta_{D_i}(\tau) > 0$  for  $i = 1$  or  $2$  in contradiction to [Kno13, Proposition 6.5].

For  $p \neq 2, 3$  we are done. For  $p = 3$  only one more case has to be checked. But, using the exceptional self-isogeny of  $\mathbf{G}_2$ , that case can be reduced to the last case of (5.9) which has been ruled out before.

Also for  $p = 2$  only one more case remains to be checked, namely  $(\mathbf{B}_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2)$ . So assume  $X = G/H$  where  $H$  is a connected subgroup of  $G = \mathrm{SO}(5)$  with  $\Sigma(X) = \{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$ . Then both simple roots are of type (b). Thus there are two colors such that  $\delta_D$  is proportional to  $\alpha_1^\vee$  and  $\alpha_2^\vee$ , respectively. Since  $\mathcal{V}(X)$  and the two colors can be separated by a hyperplane, the space  $X$  is affine, [Kno91, Thm. 6.7], and  $H$  is reductive. Since there are only two colors,  $H$  is even semisimple ([Kno13, Proposition 2.1]). Moreover,  $\dim H = 4$  (see [Kno91, Thm. 6.6]). But such a group does not exist.  $\square$

**Remark.** Let  $\alpha \in S \cap \frac{1}{2}\Sigma(X)$ . Then Luna's axiom  $(\Sigma 1)$  for a spherical system stipulates that  $\langle \sigma, \alpha^\vee \rangle$  is a nonpositive even integer for all  $\sigma \in \Sigma(X) \setminus \{2\alpha\}$  (see [Lun01, 2.1] or [BL11, 1.2.1]). The discussion above shows that the nonnegativity part is in fact superfluous since it follows from the other axioms (more precisely from (S), the parity part of  $(\Sigma 1)$ , and  $(\Sigma 2)$ ).

In passing, we proved most parts of the following statement which we state for future reference:



**5.4. Proposition.** *Let  $\sigma \neq \tau \in \Sigma(X)$  with  $|\sigma| \subseteq |\tau|$ . Then  $(|\tau|, \sigma, \tau)$  is contained in the following table:*

	$ \tau $	$\sigma$	$\tau$
	$B_4$	$\alpha_2 + 2\alpha_3 + 3\alpha_4$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$
	$C_n, n \geq 2$	$\alpha_1$	$\alpha_1 + 2\alpha_2 + \dots + \alpha_n$
	$C_n, n \geq 2$	$2\alpha_1$ ( $p \neq 2$ )	$\alpha_1 + 2\alpha_2 + \dots + \alpha_n$
	$G_2$	$\alpha_1$	$\alpha_1 + \alpha_2$
	<i>Additionally for <math>p = 2</math>:</i>		
(5.10)	$A_2$	$\alpha_1$	$\alpha_1 + \alpha_2$
	$B_n, n \geq 2$	$\alpha_1$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n$
	$B_n, n \geq 2$	$\alpha_2 + \dots + \alpha_n$	$\alpha_1 + 2\alpha_2 + \dots + 2\alpha_n$
	$C_n, n \geq 2$	$2\alpha_2 + \dots + \alpha_n$	$\alpha_1 + 2\alpha_2 + \dots + \alpha_n$
	$C_4$	$2\alpha_2 + 4\alpha_3 + 3\alpha_4$	$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$
	$G_2$	$\alpha_2$	$\alpha_1 + \alpha_2$
	<i>Additionally for <math>p = 3</math>:</i>		
	$G_2$	$\alpha_2$	$3\alpha_1 + \alpha_2$

*Proof.* Most cases are already contained in table (5.5). Missing from that table are those cases with  $\#|\tau| = 2$  which can be easily dealt with by hand. So it remains to be shown that the case  $(|\tau|, \tau, \sigma) = (C_3, 2\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_2 + \alpha_3)$  (and the isogenous case  $(B_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3)$ ) with  $p = 2$  does not occur.

By Lemma 5.3 we may localize at  $\Sigma$  (see [Kno13, 6.1]). Thus, it suffices to exclude the existence of a connected subgroup  $H \subseteq G = \mathrm{Sp}(6)$  with  $\Sigma(X) = \{\sigma, \tau\}$ . There are two colors  $D_1$  and  $D_2$  which are both of type (b). Hence  $\delta_{D_i}$  is a positive multiple of  $\alpha_i^r$ . The affinity criterion in [Kno91] shows that  $H$  is reductive. Then [Kno13, 2.1] implies that  $H$  is even semisimple. Moreover, from  $S^{(p)} = \{\alpha_2\}$  one computes  $\dim H = 11$ . Hence,  $H$  must be of type  $A_1 \times A_2$ . Since  $\mathrm{rk} H = \mathrm{rk} G = 3$  the root system of  $H$  would be a subroot system of  $G$  which is not the case.  $\square$

**Remark.** All cases occur. For the first four see, e.g., [Was96]. The others are treated in the remark after Theorem 4.5 or are isogenous to one of the previous cases.

## 6. Invariant valuations of arbitrary $G$ -varieties

For  $\mathrm{char} k = 0$ , Brion's Theorem on the valuation cone of spherical varieties has been used in [Kno93] to obtain generalizations for arbitrary  $G$ -varieties. Now that we have similar results at our disposal we can do the same in arbitrary characteristic.

The idea is to consider everything “relative” to  $B$ -invariants. More precisely, let  $X$  be any  $G$ -variety and let  $K = k(X)^B$  be its field of  $B$ -invariant rational functions. Clearly,  $K = k$  means that  $X$  is spherical. In general, the transcendence degree of  $K$  over  $k$  is called the *complexity* of  $X$  so that spherical means complexity 0.

Let  $\mathcal{V}(X)$  be the set of  $G$ -invariant ( $\mathbb{Q}$ -valued) valuations of the field  $k(X)$ . This set plays the same rôle in determining equivariant compactifications of  $X$  as in the spherical case (see [LV83]). Because it is, in general, too big to control, we partition it into manageable pieces by fixing the restriction to  $K$ . More precisely, for a valuation  $v_0$  of  $K$  put

$$(6.1) \quad \mathcal{V}_{v_0}(X) := \{v \in \mathcal{V}(X) \mid v|_K = v_0\}.$$

It turns out (see below) that these sets are quite easy to understand. So if it is possible to control all valuations of  $K$  then it is possible to understand  $\mathcal{V}(X)$  as a whole. This is trivially the case for  $K = k$  (i.e., spherical varieties) but also if  $K$  is of transcendence degree 1 over  $k$  (the case of complexity 1).

Of particular importance are *central valuations*, i.e., valuations whose restriction to  $K$  is trivial. These are precisely the elements of  $\mathcal{V}_0(X)$ . Now consider the group  $k(X)^{(B)}$  of  $B$ -semiinvariant rational functions on  $X$  and let  $\Xi(X) \subseteq \Xi(T)$  be the group of ensuing characters. Then we obtain a short exact sequence

$$(6.2) \quad 1 \rightarrow K^* \rightarrow k(X)^{(B)} \rightarrow \Xi(X) \rightarrow 0.$$

A central valuation  $v$  is by definition trivial on  $K^*$ . Hence it defines a homomorphism on  $\Xi(X)$ . This way we get a map

$$(6.3) \quad \mathcal{V}_0(X) \rightarrow N_{\mathbb{Q}}^0(X) := \text{Hom}(\Xi(X), \mathbb{Q})$$

which turns out to be injective ([Kno93, 3.6]). It identifies  $\mathcal{V}_0(X)$  with a finitely generated convex cone ([Kno93, 6.5]). The generalization of Corollary 4.7 is:

**6.1. Theorem.** *Let  $X$  be a  $G$ -variety defined over a field of characteristic  $p \neq 2$ . Then the set  $\mathcal{V}_0(X)$  of central valuations is the fundamental domain for a finite reflection group  $W_X$  acting on  $\Xi_p(X)$ .*

*Proof.* Follows from the spherical case in the same way as in characteristic zero ([Kno93, 9.2]). See also the slightly overoptimistic remark after the theorem.  $\square$

Surprisingly, this determines also the structure of non-central valuations. They have to satisfy a mild technical condition, though: A valuation  $v$  of  $k(X)$  is called *geometric* if there exists a normal model  $\overline{X}$  of  $X$  and an irreducible divisor  $D$  in  $\overline{X}$  such that  $v$  is a rational multiple of the induced valuation  $v_D$ . It is known that an invariant valuation is geometric if and only if its restriction to  $K$  is geometric ([Kno93, 4.4]). Since valuations of fields of transcendence degree  $\leq 1$  are always geometric, geometricity is no restriction for varieties of complexity  $\leq 1$ .

Now fix a geometric valuation  $v_0$  and let  $N_{\mathbb{Q}}^{v_0}(X)$  be the set of homomorphisms  $v : k(X)^{(B)} \rightarrow \mathbb{Q}$  such that  $v|_{K^*} = v_0$ . Any two elements differ by an element of  $N_{\mathbb{Q}}^0(X)$ . This means that  $N_{\mathbb{Q}}^{v_0}(X)$  has the structure of an affine space with  $N_{\mathbb{Q}}^0(X)$  as group of translations. As before, the map  $\mathcal{V}_{v_0}(X) \rightarrow N_{\mathbb{Q}}^{v_0}(X)$  is injective, thereby identifying  $\mathcal{V}_{v_0}(X)$  with a locally polyhedral convex set.

**6.2. Theorem.** *Let  $X$  be a  $G$ -variety defined over a field of characteristic  $p \neq 2$  and let  $v_0$  be a geometric valuation of  $K = k(X)^B$ . Then  $\mathcal{V}_{v_0}(X) = \tilde{v} + \mathcal{V}_0(X)$  for some  $\tilde{v} \in \mathcal{V}_{v_0}(X)$ . In particular, there is an action of  $W_X$  on  $N_{\mathbb{Q}}^{v_0}(X)$  generated by affine linear reflections such that  $\mathcal{V}_{v_0}(X)$  is a fundamental domain for this action.*

*Proof.* Again, the same proof ([Kno93, 9.2]) as in characteristic zero works. The  $W_X$ -action is defined by  $w(\tilde{v} + v) = \tilde{v} + wv$  where  $v \in \mathcal{V}_0(X)$ .  $\square$

**Remark.** Observe that this action does not depend on the choice of  $\tilde{v}$  because any two choices differ by a  $W_X$ -fixed vector.

Now we glue everything together. For this let

$$(6.4) \quad \mathcal{V}^{\text{geom}}(X) = \{v \in \mathcal{V}(X) \mid v \text{ geometric}\}$$

Moreover, it can be shown ([Kno93, 4.3]) that  $\bigcup_{v_0 \text{ geom.}} N_{\mathbb{Q}}^{v_0}(X)$  equals the set  $\mathcal{V}^{\text{geom}}(k(X)^U; T)$  of  $T$ -invariant geometric valuations of the field of  $U$ -invariant rational functions on  $X$  (with  $U = (B, B)$ ).

**6.3. Corollary.** *Let  $p \neq 2$ . Then there is a  $W_X$ -action on  $\mathcal{V}^{\text{geom}}(k(X)^U; T)$ , acting as an affine linear reflection group on each piece  $N_{\mathbb{Q}}^{v_0}(X)$ , such that  $\mathcal{V}^{\text{geom}}(X)$  is a fundamental domain for this action.*

## 7. Table of cuspidal spherical varieties of rank 1 for groups of adjoint type

$G$	$H$		$\sigma, S^{(p)}$
$\mathrm{PGL}(2)$	$\langle s_{\alpha_1} \rangle \cdot \mathrm{GL}(1)$		$\overset{1}{\circ} [\times 2]_{p \neq 2}$
$\mathrm{PGL}(n)$	$\mathrm{GL}(n-1)$	$n \geq 3$	$\overset{1}{\circ} - \overset{1}{\bullet} - \overset{1}{\bullet} - \dots - \overset{1}{\bullet} - \overset{1}{\circ}$
$\mathrm{PGL}(4)$	$\mathrm{PSp}(4)$		$\overset{1}{\bullet} - \overset{2}{\circ} - \overset{1}{\bullet}$
$\mathrm{SO}(2n+1)$	$\langle s_{\alpha_n} \rangle \cdot \mathrm{SO}(2n)$	$n \geq 2$	$\overset{1}{\circ} - \overset{1}{\bullet} - \overset{1}{\bullet} - \dots - \overset{1}{\bullet} \Rightarrow \overset{1}{\bullet} [\times 2]_{p \neq 2}$
$\mathrm{SO}(2n+1)$	$P_n(\mathrm{SO}(2n))$	$n \geq 2$	$\overset{1}{\circ} - \overset{1}{\bullet} - \overset{1}{\bullet} - \dots - \overset{1}{\bullet} \Rightarrow \overset{1}{\circ}$
$\mathrm{SO}(7)$	$\mathbf{G}_2$		$\overset{1}{\bullet} - \overset{2}{\bullet} \Rightarrow \overset{3}{\circ}$
$\mathrm{PSp}(2n)$	$\mathrm{Sp}(2) \cdot \mathrm{Sp}(2n-2)$	$n \geq 3$	$\overset{1}{\bullet} - \overset{2}{\circ} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \Leftarrow \overset{1}{\bullet}$
$\mathrm{PSp}(2n)$	$P_1(\mathrm{Sp}(2)) \cdot \mathrm{Sp}(2n-2)$	$n \geq 3$	$\overset{1}{\circ} - \overset{2}{\circ} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \Leftarrow \overset{1}{\bullet}$
$\mathrm{PSO}(2n)$	$\mathrm{SO}(2n-1)$	$n \geq 4$	$\overset{2}{\circ} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \begin{matrix} \nearrow \overset{1}{\bullet} \\ \searrow \overset{1}{\bullet} \end{matrix}$
$\mathbf{F}_4$	$\mathrm{Spin}(9)$		$\overset{1}{\bullet} - \overset{2}{\bullet} \Rightarrow \overset{3}{\bullet} - \overset{2}{\circ}$
$\mathbf{G}_2$	$\langle s_{\alpha_1} \rangle \cdot \mathrm{SL}(3)$		$\overset{2}{\circ} \Leftarrow \overset{1}{\bullet} [\times 2]_{p \neq 2}$
$\mathbf{G}_2$	$\mathrm{GL}(2)_{\mathrm{long}} U_{2\alpha_1+\alpha_2} U_{3\alpha_1+\alpha_2} U_{3\alpha_1+2\alpha_2}$		$\overset{1}{\circ} \Leftarrow \overset{1}{\circ}$
$\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ $(\mathrm{id} \times F_q) \mathrm{PGL}(2)$ $F_q = \text{Frobenius morphism}, q = p^l \geq 1$			$\overset{q}{\circ} \begin{matrix} \text{---} \end{matrix} \overset{1}{\circ}$
Additionally for $p = 2$ :			
$\mathrm{SO}(2n+1)$	$\mathrm{SO}(3) \times \mathrm{SO}(2n-1)$	$n \geq 3$	$\overset{1}{\bullet} - \overset{2}{\circ} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \Rightarrow \overset{2}{\bullet}$
$\mathrm{SO}(2n+1)$	$P_1(\mathrm{SO}(3)) \times \mathrm{SO}(2n-1)$	$n \geq 3$	$\overset{1}{\circ} - \overset{2}{\circ} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \Rightarrow \overset{2}{\bullet}$
$\mathrm{PSp}(2n)$	$\langle s_{\alpha_n} \rangle \cdot \mathrm{PSO}(2n)$	$n \geq 2$	$\overset{2}{\circ} - \overset{2}{\bullet} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \Leftarrow \overset{1}{\bullet}$
$\mathrm{PSp}(2n)$	$P_n(\mathrm{PSO}(2n))$	$n \geq 2$	$\overset{2}{\circ} - \overset{2}{\bullet} - \overset{2}{\bullet} - \dots - \overset{2}{\bullet} \Leftarrow \overset{1}{\circ}$
$\mathrm{PSp}(6)$	$\mathbf{G}_2$		$\overset{2}{\bullet} - \overset{4}{\bullet} \Leftarrow \overset{3}{\circ}$
$\mathbf{F}_4$	$\mathrm{Sp}(8)$		$\overset{2}{\circ} - \overset{3}{\bullet} \Rightarrow \overset{4}{\bullet} - \overset{2}{\bullet}$
Additionally for $p = 3$ :			
$\mathbf{G}_2$	$\langle s_{\alpha_2} \rangle \cdot \mathrm{SL}(3)_{\mathrm{short}}$		$\overset{3}{\bullet} \Leftarrow \overset{2}{\circ} [\times 2]$
$\mathbf{G}_2$	$\mathrm{GL}(2)_{\mathrm{short}} U_{\alpha_1+\alpha_2} U_{2\alpha_1+\alpha_2} U_{3\alpha_1+2\alpha_2}$		$\overset{3}{\circ} \Leftarrow \overset{1}{\circ}$

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